Proof of Brioschi's Recursion Formula for the Expansion of the Even σ-Functions of Two Variables.

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In a note published in the Goettinger Nachrichten for 1890, p. 237, Brioschi has given, without proof, a recursion formula for the expansion of the even σ-functions of two variables, in which he makes use of a peculiar differential operator considerably easier to handle than the Aronhold process used by Wiltheiss* for the same purpose. He also gives the results of the application of this operator to the simultaneous concomitants of two cubics, and thus furnishes everything that is necessary for the actual computation of the successive terms of the expansion of the even σ-functions of two variables into power series.

In the following pages I propose to give a proof of these theorems of Brioschi's, since, as far as I know, no proof of them has ever been published.

§1. *The Partial Differential Equations for the Even σ-Functions of Two Variables.*

Let†

$$R(x) = \alpha_x^6 = \beta_x^6 = \ldots . \tag{1}$$

be a (non-homogeneous) sextic, and

$$H = (\alpha\beta)^2 \alpha_x^4 \beta_x^4, \quad i = (\alpha\beta)^4 \alpha_x^2 \beta_x^2, \quad A = (\alpha\beta)^6; \tag{2}$$

let, further,

$$R = \phi \cdot \psi \tag{3}$$

be one of the ten possible decompositions of R into two cubic factors, and

$$\vartheta = (\phi, \psi)_1, \quad \Theta = (\phi, \psi)_2, \quad J = (\phi, \psi)_3, \tag{4}$$

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* Math. Annalen, Bd. 29, p. 272; Bd. 35, p. 433; Bd. 36, p. 134.

† Since we are using non-homogeneous variables, the symbols α_x, β_x, etc., stand for $\alpha_1 x + \alpha_2$, $\beta_1 x + \beta_2$, etc.

and write

$$\phi = \phi_0 x^3 + 3\phi_1 x^2 + 3\phi_2 x + \phi_3, \quad \psi = \psi_0 x^3 + 3\psi_1 x^2 + 3\psi_2 x + \psi_3$$
$$\vartheta = \vartheta_0 x^4 + 4\vartheta_1 x^3 + 6\vartheta_2 x^2 + 4\vartheta_3 x + \vartheta_4 .$$

Let, further,

$$y_2 = R(x) \tag{5}$$

and choose

$$w_1 = \int \frac{xdx}{2y}, \quad w_2 = \int \frac{dx}{2y} \tag{6}$$

for the two integrals of the first kind used in the formation of the σ-functions.

If, then, a denote one of the roots of ϕ, the following theorem holds:

Theorem I.

The σ-function belonging to the decomposition (3) *satisfies the partial differential equation*

$$\frac{\partial \sigma}{\partial a} = -\tfrac{1}{2}\sigma . \sum_{\alpha, \beta} \lambda_{\alpha\beta} u_\alpha u_\beta - \tfrac{9}{20} \frac{\Theta(a)}{R'(a)} \sigma - \sum_{\alpha, \beta} \varkappa_{\alpha\beta} u_\alpha \frac{\partial \sigma}{\partial u_\beta} + \tfrac{1}{4} \sum_{\alpha, \beta} \frac{\partial^2 \sigma}{\partial u_\alpha \partial u_\beta} \frac{a^{4-\alpha-\beta}}{R'(a)}, \tag{A}$$

in which the summation indices take independently the values 1, 2 *and the quantities* $\lambda_{\alpha\beta}$, $\varkappa_{\alpha\beta}$ *are defined by the equations*

$$R'(a)(\lambda_{11} x^2 + 2\lambda_{12} x + \lambda_{22}) = -\tfrac{3}{4} i_x^2 i_a^2 - \tfrac{3}{40} A (x-a)^2 \tag{7}$$

$$\begin{aligned} &R'(a)(\varkappa_{11} x - \varkappa_{12} x\xi + \varkappa_{21} - \varkappa_{22}\xi) \\ &= \tfrac{3}{2} \frac{\vartheta_0}{\phi_0} (3\phi_1 a^2 + 3\phi_2 a + \phi_3)(x-\xi) - \tfrac{3}{2}(3\vartheta_1 a^2 + 3\vartheta_2 a + \vartheta_3)(x-\xi) \\ &\qquad + \tfrac{3}{2} \vartheta_a^2 \vartheta_x \vartheta_\xi + \tfrac{9}{10} \Theta_x \Theta_a (a-\xi) + \tfrac{1}{4} J (x-a)(\xi-a). \end{aligned} \tag{8}$$

Proof: In a previous paper, "The Partial Differential Equations of the Hyperelliptic Θ- and σ-Functions,"* I have given a proof of (A) with the following defining equations for the $\varkappa_{\alpha\beta}$'s and $\lambda_{\alpha\beta}$'s:

$$\begin{aligned} \Lambda(x, \xi) &\equiv \tfrac{1}{4} [\lambda_{11} x . + \lambda_{12}(x+\xi) + \lambda_{22}] \\ &= \tfrac{1}{4} \left(\frac{1}{x-a} + \frac{1}{\xi - a} \right) \frac{F(x, \xi)}{(x-\xi)^2} - \tfrac{1}{2} \frac{F(x, a) F(\xi, a)}{R'(a)(x-a)^2(\xi-a)^2} + \tfrac{1}{2} \frac{\frac{\partial}{\partial a} F(x, \xi)}{(x-\xi)^2} \end{aligned} \tag{9}$$

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* American Journal of Mathematics, vol. XXI, p. 107, equations.

and

$$K(x,\, \xi) \equiv \varkappa_{11}x - \varkappa_{12}x\xi + \varkappa_{21} - \varkappa_{22}\xi = \tfrac{1}{2}\,\frac{(x-\xi)}{x-a} - \frac{(a-\xi)\,F(x,\, a)}{R'(a)(x-a)^2}\,, \qquad (10)$$

where $F(x,\, \xi) = \alpha_x^3\alpha_\xi^3$ is the third polar of $R(x)$ with respect to ξ.

It only remains therefore to show that (9) and (10) are equivalent to (7) and (8).

a). The expression for $\Lambda(x,\, \xi)$ can be transformed as follows:

Notice first that $\Lambda(x,\, \xi)$ is the first polar of $\Lambda(x,\, x)$ with respect to ξ; but by letting $\xi = x$ and making use of the expansion

$$F(x,\, \xi) = R(x) + \tfrac{1}{2}R'(x)(\xi - x) + \tfrac{1}{10}R''(x)(\xi - x)^2 + \ldots.$$

we obtain

$$R'(a)\Lambda(x,\, x) = -\tfrac{1}{2}\left(\frac{F(x,\, a)}{(x-a)^2}\right)^2 + \tfrac{3}{20}\,\frac{R(x)\,R'(a) - R(a)\,R'(x)}{(x-a)^3} - \tfrac{1}{40}\,\frac{R'(a)\,R'(x)}{(x-a)^2}\,,$$

where the zero term $R(a)\,R'(x)$ has been added for symmetry.

But since

$$R(x) = \alpha_x^5\alpha_1 \,.\, x + \alpha_x^5\alpha_2 = \tfrac{1}{6}\,xR'(x) + \alpha_x^5\alpha_2,$$

we have

$$R(x)\,R'(a) - R(a)\,R'(x) = \tfrac{1}{6}\,(x-a)\,R'(x)\,R'(a) - 6\,(\alpha\beta)\,\alpha_x^5\beta_a^5,$$

and therefore

$$R'(a)\Lambda(x,\, x) = -\frac{1}{(x-a)^4}\left[\tfrac{1}{2}\,\alpha_x^3\alpha_a^3\,.\,\beta_x^3\beta_a^3 + \tfrac{9}{10}\,(x-a)\,.\,(\alpha\beta)\,\alpha_x^5\beta_a^5\right].$$

Now observe that

$$2\alpha_x^3\alpha_a^3\,.\,\beta_x^3\beta_a^3 = \alpha_x^6\beta_a^6 + \alpha_a^6\beta_x^6 - (\alpha_x^3\beta_a^3 - \alpha_a^3\beta_x^3)^2$$

and apply Clebsch–Gordon's expansion, which furnishes:

$$\begin{aligned}
&(\alpha\beta)^2(\alpha_x^4\beta_a^4 + \alpha_a^4\beta_x^4)\\
&\qquad = 2H_x^4H_a^4 + \tfrac{24}{7}\,i_x^2 i_a^2\,(x-a)^2 + \tfrac{2}{5}\,A\,(x-a)^4\\
&\qquad = 2\,(\alpha\beta)^2(\alpha_x^3\alpha_a\beta_x\beta_a^3 + \alpha_a^3\alpha_x\beta_a\beta_x^3)\;4\;H_x^4H_a^4 + \tfrac{6}{7}\,i_x^2 i_a^2\,(x-a)^2 - \tfrac{1}{5}\,A\,(x-a)^4\\
&\qquad = (\alpha\beta)\,\alpha_x^5\beta_a^5\;\tfrac{5}{2}\,H_x^4H_a^4\,(x-a) + \tfrac{25}{14}\,i_x^2 i_a^2\,(x-a)^3 + \tfrac{1}{6}\,A\,(x-a)^5.
\end{aligned}$$

The result is equation (7).

b). Similarly the expression for $K(x,\, \xi)$ can be transformed as follows:
By Clebsch–Gordon's expansion we have

$$\phi_a^3\psi_x^3 = \alpha_x^3\alpha_a^3 - \tfrac{3}{2}\,\vartheta_x^2\vartheta_a^2\,(x-a) + \tfrac{9}{10}\,\Theta_x\Theta_a\,(x-a)^2 - \tfrac{1}{4}\,J\,(x-a)^3.$$

Hence since

$$\phi_a^3 = 0,$$

$$\frac{F(x, a)}{x - a} = \tfrac{3}{2}\vartheta_x^2\vartheta_a^2 - \tfrac{9}{10}\Theta_x\Theta_a(x-a) + \tfrac{1}{4}J(x-a)^2.$$

Further,

$$\frac{\phi(x)\psi(a)}{x-a} = \frac{\phi(x)\psi(a) - \phi(a)\psi(x)}{x-a} = 3\vartheta_x^2\vartheta_a^2 + \tfrac{1}{2}J(x-a)^2,$$

and by making $x = a$:

$$R'(a) = \phi'(a)\psi(a) = 3\vartheta_a^4.$$

Thus we obtain

$$R'(a)K(x, \xi) \\ = \tfrac{3}{2}\frac{(x-\xi)\vartheta_a^4 - (a-\xi)\vartheta_a^2\vartheta_x^2}{x-a} + \tfrac{9}{10}\Theta_x\Theta_a(a-\xi) + \tfrac{1}{4}J(x-a)(\xi-a),$$

which, after performing the division by $x - a$, reduces to

$$R'(a)K(x, \xi) \\ = -\tfrac{3}{2}\vartheta_a^3\vartheta_1(x-\xi) + \tfrac{3}{2}\vartheta_a^2\vartheta_x\vartheta_\xi + \tfrac{9}{10}\Theta_x\Theta_a(a-\xi) + \tfrac{1}{4}J(x-a)(\xi-a).$$

And if we reduce the degree of the right-hand side in a by means of the equation $\phi(a) = 0$, we obtain (8).

Thus Theorem I is proved.

If b denote a root of ψ, the expression for $\dfrac{\partial\sigma}{\partial b}$ can be derived from (A) by simply writing b instead of a and interchanging ϕ and ψ, which operation changes the signs of ϑ and J, but leaves Θ, i, A unchanged. We shall refer to the differential equation thus obtained as equation (A′).

§2.—*Brioschi's Differential Operator and the Recursion Formula for the Expansion of* $\sigma_{\phi\psi}(u_1, u_2)$.

Multiply equation (A) by

$$3\vartheta_u^2\vartheta_a^2 + \tfrac{1}{2}J(u-a)^2 = \frac{\phi(u)\psi(a)}{u-a},$$

u being an arbitrary new variable, and sum with respect to the three roots of ϕ (notation Σ). Similarly, multiply (A′) by

$$-3\vartheta_u^2\vartheta_b^2 - \tfrac{1}{2}J(u-b)^2 = \frac{\psi(u)\phi(b)}{u-b}$$

and sum with respect to the three roots of ψ (notation Σ'). Add the two equations thus obtained and finally put $u = \dfrac{u_1}{u_2}$ and multiply by u_2^2.

The left-hand side becomes*

$$(3\vartheta_u^2\vartheta_1^2 + \tfrac{1}{2}Ju_2^2)\left(\Sigma a^2 \frac{\partial\sigma}{\partial a} - \Sigma' b^2 \frac{\partial\sigma}{\partial b}\right) + 2(3\vartheta_u^2\vartheta_1\vartheta_2 - \tfrac{1}{2}Ju_1u_2)\left(\Sigma a \frac{\partial\sigma}{\partial a} - \Sigma' b \frac{\partial\sigma}{\partial b}\right)$$
$$+ (3\vartheta_u^2\vartheta_2^2 + \tfrac{1}{2}Ju_1^2)\left(\Sigma \frac{\partial\sigma}{\partial a} - \Sigma' \frac{\partial\sigma}{\partial b}\right).$$

The right-hand side of (A) is an integral function of the second degree of a, say $q(a)$, divided by $R'(a)$, and the right-hand side of (A′) is an integral function of the second degree of b, say $\bar{q}(b)$, divided by $R'(b)$. But

$$\Sigma \frac{\phi(u)\psi(a)}{u-a} \frac{q(a)}{R'(a)} = \Sigma \frac{q(a)}{\phi'(a)} \frac{\phi(u)}{u-a} = q(u),$$
$$\Sigma' \frac{\psi(u)\phi(b)}{u-b} \cdot \frac{\bar{q}(b)}{R'(b)} = \Sigma \frac{\bar{q}(b)}{\psi'(b)} \frac{\psi(u)}{u-b} = \bar{q}(u).$$

Hence the result of the above-described operation is, for the right-hand side,

$$q(u) + \bar{q}(u),$$

and if we observe that†

$$\phi_0\psi_1 - \phi_1\psi_0 = \vartheta_0,$$
$$\phi_0\psi_2 - \phi_2\psi_0 = 2\vartheta_1,$$
$$\phi_0\psi_3 - \phi_3\psi_0 = 3\vartheta_2 + \tfrac{1}{2}J,$$

and

$$\frac{3\vartheta_0}{\phi_0\psi_0} = s - s',$$

where

$$s = \Sigma a, \quad s' = \Sigma' b, \tag{11}$$

this reduces to

$$3i_u^4 \, . \, \sigma - \tfrac{9}{10}\Theta_u^2 \, . \, \sigma - \tfrac{9}{5}\Theta_u^2\left(u_1 \frac{\partial\sigma}{\partial u_1} + u_2 \frac{\partial\sigma}{\partial u_2}\right)$$
$$+ \tfrac{1}{2}\left(\frac{\partial^2\sigma}{\partial u_1^2}u_1^2 + 2\frac{\partial^2\sigma}{\partial u_1 \partial u_2}u_1u_2 + \frac{\partial^2\sigma}{\partial u_2^2}u_2^2\right)$$
$$+ \tfrac{1}{2}(s - s')(3\vartheta_u^2\vartheta_1^2 + \tfrac{1}{2}J)\left(u_1\frac{\partial\sigma}{\partial u_1} + u_2\frac{\partial\sigma}{\partial u_2}\right).$$

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*It is hardly necessary to say that here ϑ_1, ϑ_2 are symbols, and that $\vartheta_u = \vartheta_1 u_1 + \vartheta_2 u_2$.

†Here ϑ_0, ϑ_1, ϑ_2 are of course coefficients of ϑ.

Transposing the last term, we obtain the

Theorem II.

If the three operators $G_0(f),\quad G_1(f),\quad G_2(f)$
are defined by the equations

$$\left.\begin{aligned} G_0(f) &= \quad \Sigma \frac{\partial f}{\partial a} - \Sigma' \frac{\partial f}{\partial b}, \\ G_1(f) &= \Sigma a \frac{\partial f}{\partial a} - \Sigma' b \frac{\partial f}{\partial b}, \\ G_2(f) &= \Sigma a^2 \frac{\partial f}{\partial a} - \Sigma' b^2 \frac{\partial f}{\partial b} - \frac{s-s'}{2}\left(u_1 \frac{\partial f}{\partial u_1} + u_2 \frac{\partial f}{\partial u_2}\right), \end{aligned}\right\} \tag{12}$$

where $s = \Sigma a, \quad s' = \Sigma' b,$

and the operator $D(f)$ *by*

$$D(f) = (6\vartheta_u^2 \vartheta_1^2 + J u_2^2)\, G_2(f) + 2(6\vartheta_u^2 \vartheta_1 \vartheta_2 - J u_1 u_2)\, G_1(f) \\ + (6\vartheta_u^2 \vartheta_2^2 + J u_1^2)\, G_0(f), \tag{13}$$

the function $\sigma_{\phi\psi}(u_1 u_2)$ *satisfies the partial differential equation*

$$D(\sigma) = 6 i_u^4 . \sigma - \tfrac{9}{5} \Theta_u^2 . \sigma - \tfrac{18}{5} \Theta_u^2 \left(u_1 \frac{\partial \sigma}{\partial u_1} + u_2 \frac{\partial \sigma}{\partial u_2}\right) \\ + \left(u_1^2 \frac{\partial^2 \sigma}{\partial u_1^2} + 2 u_1 u_2 \frac{\partial^2 \sigma}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 \sigma}{\partial u_2^2}\right). \tag{B}$$

We have chosen the two integrals of the first kind, w_1, w_2, in accordance with Wiltheiss and Brioschi; if, instead, we had taken $\bar{w}_1 = c w_1$, $\bar{w}_2 = c w_2$, we would have reached a slightly different result, viz. the first and last terms on the right would be changed into

$$\frac{1}{c^2} . 6 i_u^4 \sigma \text{ and } c^2 . \left(u_1^2 \frac{\partial^2 \sigma}{\partial u_1^2} + 2 u_1 u_2 \frac{\partial^2 \sigma}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 \sigma}{\partial u_2^2}\right),$$

as follows from the formulæ* for the passage from one canonical system of integrals to another. Klein, in his paper on "Hyperelliptische Sigmafunctionen" (Math. Annalen, Bd. 27), chooses

$$\bar{w}_1 = -\int \frac{x\,dx}{y}, \quad \bar{w}_2 = -\int \frac{dx}{y},$$

hence $c = -2$.

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* Bolza, "On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind," Chicago International Mathematical Congress Papers, p. 8.

Now let
$$\sigma(u_1, u_2) = 1 + \frac{S_1}{2!} + \frac{S_2}{4!} + \frac{S_3}{6!} + \ldots. \tag{14}$$

be the expansion of $\sigma_{\phi\psi}$ according to powers of u_1, u_2, $\frac{S_n}{2n!}$ denoting the aggregate of the terms of dimension $2n$.

Substitute this series in (B) and equate the terms of dimension $2n$ on both sides. Remembering that

$$u_1 \frac{\partial S_{n-1}}{\partial u_1} + u_2 \frac{\partial S_{n-1}}{\partial u_2} = (2n-2)\, S_{n-1}$$

and
$$u_1^2 \frac{\partial^2 S_n}{\partial u_1^2} + 2u_1 u_2 \frac{\partial^2 S_n}{\partial u_1 \partial u_2} + u_2^2 \frac{\partial^2 S_n}{\partial u_2^2} = 2n\,.\,(2n-1)\, S_n,$$

we obtain
$$D(S_{n-1}) = 12\,(n-1)(2n-3)\, i_u^4 S_{n-2} - \tfrac{9}{5}\,(4n-3)\,\Theta_u^2 S_{n-1} + S_n\,.$$

This furnishes for $n = 1$
$$S_1 = \tfrac{9}{5}\,\Theta_u^2 \tag{15}$$

and thus we reach Brioschi's theorem:

Theorem III.

The successive terms of the expansion of $\sigma_{\phi\psi}(u_1, u_2)$ *into a power series*

$$\sigma_{\phi\psi}(u_1, u_2) = 1 + \frac{S_1}{2!} + \frac{S_2}{4!} + \frac{S_3}{6!} + \ldots.$$

are determined by the recursion formula

$$S_n = D(S_{n-1}) + (4n-3)\, S_1 S_{n-1} - 12\,(n-1)(2n-3)\, i_u^4 S_{n-2} \tag{C}$$

where the operator D is defined by (13) *and*

$$S_1 = \tfrac{9}{5}\,(\phi\psi)^2\, \phi_u \psi_u\,, \quad i_u^4 = (\alpha\beta)^4\, \alpha_u^2 \beta_u^2\,.$$

Corollary: The covariant i is expressible in terms of the simultaneous concomitants of ϕ and ψ as follows:*

$$i = \tfrac{6}{25}\,\Theta^2 + \tfrac{3}{10}\,\Delta\nabla - \tfrac{1}{5}\,J\vartheta\,, \tag{16}$$

in the notation of Clebsch, "Binaerie Formen," §61.

* Proved by Càporali, Sul sistema di due forme binarie cubiche, §8, Rendiconto della R. Accademia di Napoli, 1883.

§3. *Reduction of Brioschi's Operator $D(f)$ to an Aronhold Process.*

From

$$3\frac{\partial\phi_1}{\partial a} = -\phi_0$$
$$3\frac{\partial\phi_2}{\partial a} = -(\phi_0 a + 3\phi_1)$$
$$\frac{\partial\phi_3}{\partial a} = -(\phi_0 a^2 + 3\phi_1 a + 3\phi_2)$$

follows

$$\Sigma \frac{\partial f}{\partial a} = -\phi_0\frac{\partial f}{\partial\phi_1} - 2\phi_1\frac{\partial f}{\partial\phi_2} - 3\phi_2\frac{\partial f}{\partial\phi_3}$$
$$\Sigma a \frac{\partial f}{\partial a} = \phi_1\frac{\partial f}{\partial\phi_1} + 2\phi_2\frac{\partial f}{\partial\phi_2} + 3\phi_3\frac{\partial f}{\partial\phi_3}$$
$$\Sigma a^2 \frac{\partial f}{\partial a} = 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_2} - \frac{3\phi_1}{\phi_0}\left(\phi_0\frac{\partial f}{\partial\phi_0} + \phi_1\frac{\partial f}{\partial\phi_1} + \phi_2\frac{\partial f}{\partial\phi_2} + \phi_3\frac{\partial f}{\partial\phi_3}\right).$$

Hence if f be a homogeneous function of $\phi_0, \phi_1, \phi_2, \phi_3$ of degree ν and of u_1, u_2 of degree m, we have

$$\Sigma a^2\frac{\partial f}{\partial a} - \frac{s}{2}\left(u_1\frac{\partial f}{\partial u_1} + u_2\frac{\partial f}{\partial u_2}\right) = s\left(\nu - \frac{m}{2}\right)f + 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_3}.$$

We are going to apply the operator D successively to $S_1, S_2, S_3, \ldots$; but it is easily seen from (C) that S_n is of degree $2n$ in u_1, u_2, of degree n in the ϕ_i's and of degree n in the ψ_i's. Hence we are only dealing with functions f for which

$$\nu - \frac{m}{2} = 0,$$

and for such functions we have

$$\Sigma a^2\frac{\partial f}{\partial a} - \frac{s}{2}\left(u_1\frac{\partial f}{\partial u_1} + u_2\frac{\partial f}{\partial u_2}\right) = 3\phi_1\frac{\partial f}{\partial\phi_0} + 2\phi_2\frac{\partial f}{\partial\phi_1} + \phi_3\frac{\partial f}{\partial\phi_2}.$$

Analogous formulæ hold for

$$\Sigma'\frac{\partial f}{\partial b},\quad \Sigma' b\frac{\partial f}{\partial b},\quad \Sigma' b^2\frac{\partial f}{\partial b}$$

and we thus obtain for $G_0(f)$, $G_1(f)$, $G_2(f)$ expressions which are *homogeneous and linear* in

$$\frac{\partial f}{\partial \phi_0},\ \frac{\partial f}{\partial \phi_1},\ \frac{\partial f}{\partial \phi_2},\ \frac{\partial f}{\partial \phi_3},\ \frac{\partial f}{\partial \psi_0},\ \frac{\partial f}{\partial \psi_1},\ \frac{\partial f}{\partial \psi_2},\ \frac{\partial f}{\partial \psi_3}.$$

For their further simplification we may therefore write symbolically:

$$\frac{\partial f}{\partial \phi_0} = x_1^3,\quad \frac{\partial f}{\partial \phi_1} = 3x_1^2 x_2,\quad \frac{\partial f}{\partial \phi_2} = 3x_1 x_2^2,\quad \frac{\partial f}{\partial \phi_3} = x_2^3$$
$$\frac{\partial f}{\partial \psi_0} = y_1^3,\quad \frac{\partial f}{\partial \psi_1} = 3y_1^2 y_2,\quad \frac{\partial f}{\partial \psi_2} = 3y_1 y_2^2,\quad \frac{\partial f}{\partial \psi_3} = y_2^3,$$

perform the reductions in the symbolical form and in the final result resubstitute $\frac{\partial f}{\partial \phi_0}$, etc., for the symbols.

Thus

$$\begin{aligned} G_0(f) &= -3\,[x_2\phi_x^2\phi_1 - y_2\psi_y^2\psi_1], \\ G_1(f) &= 3\,[x_2\phi_x^2\phi_2 - y_2\psi_y^2\psi_2], \\ G_2(f) &= 3\,[x_1\phi_x^2\phi_2 - y_1\psi_y^2\psi_2]. \end{aligned}$$

Since the degree ν of f in the ϕ_i's is supposed to be the same as in the ψ_i's, we have

$$\phi_x^3 = \psi_y^3 = \nu f,$$

and $G_1(f)$ may therefore be written

$$\begin{aligned} G_1(f) &= 3\,[\phi_x^3 - \psi_y^3 - x_1\phi_x^2\phi_1 + y_1\psi_y^2\psi_1] \\ &= 3\,[-x_1\phi_x^2\phi_1 + y_1\psi_y^2\psi_1], \end{aligned}$$

hence

$$2G_1(f) = 3\,[(x_2\phi_2 - x_1\phi_1)\,\phi_x^2 - (y_2\psi_2 - y_1\psi_1)\,\psi_y^2].$$

Now if we put

$$\Phi_z^2 = 6\vartheta_u^2\vartheta_z^2 + J(uz)^2 = c_{11}z_1^2 + 2c_{12}z_1z_2 + c_{22}z_2^2$$

we may write

$$\begin{aligned} D(f) &= c_{11}G_2(f) + 2c_{12}G_1(f) + c_{22}G_0(f) \\ &= 3\phi_x^2\,[c_{11}x_1\phi_2 + c_{12}(x_2\phi_2 - x_1\phi_1) - c_{22}x_2\phi_1] \\ &\quad - 3\psi_y^2\,[c_{11}y_1\psi_2 + c_{12}(y_2\psi_2 - y_1\psi_1) - c_{22}y_2\psi_1], \end{aligned}$$

or

$$D(f) = 3\,(\Phi\phi)\,\Phi_x\phi_x^2 - 3\,(\Phi\psi)\,\Phi_y\psi_y^2.$$

We thus reach the

Theorem IV.

If

$$\Phi_x^2 = 6\vartheta_u^2\vartheta_x^2 + J(ux)^2 \tag{17}$$

and

$$\begin{aligned} 3\,(\Phi\phi)\,\Phi_x\phi_x^2 &= M_0x_1^3 + 3M_1x_1^2x_2 + 3M_2x_1x_2^2 + M_3x_2^3 = M_x^3 \\ -3\,(\Phi\psi)\,\Phi_x\psi_x^2 &= N_0x_1^3 + 3N_1x_1^2x_2 + 3N_2x_1x_2^2 + N_3x_2^3 = N^3 \end{aligned} \tag{18}$$

then Brioschi's operator $D(f)$ is equivalent to the following Aronhold process:

$$D(f)=\sum_{i=0}^{3} M_i \frac{\partial f}{\partial \phi_i}+\sum_{i=0}^{3} N_i \frac{\partial f}{\partial \psi_i}. \tag{D}$$

Corollary: Hence follows the rule:

To obtain $D(f)$, replace in f the coefficients ϕ_i and ψ_i by the corresponding coefficients $\phi_i+\lambda M_i$ and $\psi_i+\lambda N_i$ respectively and expand according to powers of λ. $D(f)$ will be the coefficient of λ in this expansion.

In applying this rule to a function of u_1, u_2:

$$f=f_u^m,$$

we first compute

$$D(f_x^m)$$

and in the result put $x=u$. This precaution is necessary, if we wish to use symbolical methods, since u_1, u_2 enter into the coefficients of M and N as well as into the function f_u^m.

§4. *Effect of the Operator D upon the Simultaneous Concomitants of ϕ and ψ.*

We now proceed to determine the effect of the operator D upon the simultaneous concomitants of ϕ and ψ as far as they appear in the successive terms S_1, S_2, S_3, ...

1. Since

$$S_1=\tfrac{2}{5}\Theta,$$

we need, in order to obtain S_2, the value of $D(\Theta)$. According to the above rule

$$D(\Theta_x^2)=(\phi^1, N)_2+(\psi, M)_2;$$

but

$$\begin{aligned}(M, \psi)_2 &= \quad 3(\Phi\phi)(\phi\psi)^2\Phi_x\psi_x+2(\Phi\phi)^2(\phi\psi)\psi_x^2,\\ (N, \phi)_2 &= -3(\Phi\psi)(\psi\phi)^2\Phi_x\phi_x-2(\Phi\psi)^2(\psi\phi)\phi_x^2,\end{aligned}$$

therefore

$$D(\Theta_x^2)=-3J\Phi_x^2+2(\phi\psi)[(\Phi\phi)^2\psi_x^2+(\Phi\psi)^2\phi_x^2].$$

But since*

$$(\vartheta, \phi)_2=\tfrac{1}{6}J\phi, \quad (\vartheta, \phi)_3=-\tfrac{3}{4}p,$$

we have

$$\left.\begin{aligned}(\Phi\phi)^2\phi_x &= 2J\phi_u^2\phi_x-3p_u(ux),\\ (\Phi\psi)^2\psi_x &= 2J\psi_x^2\psi_u+3\pi_u(ux),\end{aligned}\right\} \tag{18}$$

* Càporali, Rend. della R. Acc. di Napoli, Marzo, 1883, §3.

hence

$$(\phi\psi)(\Phi\phi)^2\psi_x^2 = 2J(\phi\psi)\,\phi_u^2\psi_x^2 + 3p_u\cdot\psi_u\psi_x^2,$$
$$(\phi\psi)(\Phi\psi)^2\phi_x^2 = 2J(\phi\psi)\,\psi_u^2\phi_x^2 + 3\pi_u\cdot\phi_u\phi_x^2.$$

Collect the terms, apply Clebsch-Gordan's expansion and make use of the formulæ*

$$\Gamma = \tfrac{1}{2}(\phi\pi + \psi p) = J\vartheta + \Delta\nabla - \Theta^2, \tag{19}$$
$$2\Gamma_x^3\Gamma_u = \phi_x^2\phi_u\cdot\pi_x + \psi_x^2\psi_u\cdot p_x, \tag{19a}$$
$$2\Gamma_x^2\Gamma_u^2 = \phi_u\phi_x^2\cdot\pi_u + \psi_u\psi_x^2\cdot p_u, \tag{19b}$$

we obtain

$$D(\Theta_x^2) = 12\Gamma_u^2\Gamma_x^2 - 10J\vartheta_u^2\vartheta_x^2 - \tfrac{1}{3}J^2(ux)^2, \tag{20}$$

and putting $x = u$:

$$D(\Theta) = 12\Delta\nabla - 12\Theta^2 + 2J\vartheta. \tag{21}$$

The reduction formula (C) furnishes now for S_2 the value

$$S_2 = 18\Delta\nabla + 6J\vartheta - \tfrac{207}{25}\Theta^2. \tag{22}$$

2. To obtain S_3 we need $D(J)$, $D(\vartheta)$ and $D(\Delta\nabla)$.

a). $D(J) = (\phi,\, N)_3 + (M,\, \psi)_3$

$$(M,\, \psi)_3 = 3(\Phi\phi)(\Phi\psi)(\phi\psi)^2 = 3(\Theta,\, \Phi)_2.$$

But†

$$(\Theta,\, \Phi)_2 = 6(\vartheta\Theta)^2\vartheta_u^2 + J\Theta_u^2 = -3\nu - J\Theta$$

where

$$\nu = (\nabla,\, \Delta)_1.$$

Hence

$$(M,\, \psi)_3 = -9\nu - 3J\Theta$$
$$(N,\, \phi)_3 = 9\nu + 3J\Theta,$$

consequently

$$D(J) = -18\nu - 6J\Theta. \tag{23}$$

b). $D(\vartheta_x^4) = (\phi,\, N)_1 + (M,\, \psi)_1$

$$(M,\, \psi)_1 = 3(\Phi\phi)(\Phi\psi)\,\phi_x^2\psi_x^2 - 2(\Phi\phi)^2\,\phi_x\cdot\psi_x^3$$
$$= -\tfrac{1}{2}(\Phi\phi)^2\,\phi_x\cdot\psi_x^3 + \tfrac{3}{2}(\Phi\psi)^2\,\psi_x\cdot\phi_x^3 - \tfrac{3}{2}(\phi\psi)^2\,\phi_x\psi_x\cdot\Phi_x^2.$$

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* v. Gall, "Syzyganten cubischer Formen." Math. Annalen, Bd. 31, p. 435, and Gordan, "Invariantentheorie," II, p. 335.

† Càporali, l. c.

Hence putting $x=u$ and remembering that $(N, \phi)_1$ can be derived from $(M, \psi)_1$ by interchanging ϕ and ψ:

$$D(\vartheta)=4J.\phi\psi-18\Theta\vartheta. \tag{24}$$

c). $D(\Delta\nabla)=\Delta D(\nabla)+\nabla D(\Delta)$,

$$D(\Delta_x^2)=2(\phi, M)_2=6(\phi\phi')^2(\Phi\phi)\Phi_x\phi'_x+4(\phi\phi')(\Phi\phi)^2\phi'^2_x$$
$$=2(\phi\phi')^2(\Phi\phi)\Phi_x\phi'_x=2(\Phi, \Delta)_1.$$

Similarly,

$$D(\nabla_x^2)=-2(\Phi, \nabla)_1,$$

hence

$$D(\Delta_x^2\nabla_x^2)=-2[\Delta_x^2(\Phi, \nabla)_1-\nabla_x^2(\Phi, \Delta)_1]$$
$$=-2(\Delta\nabla)\Delta_x\nabla_x\,.\,\Phi_x^2. \tag{25}$$

Hence, putting $x=u$,

$$D(\Delta\nabla)=12\nu\vartheta. \tag{26}$$

The reduction formula (C) furnishes now for S_3 the value:

$$S_3=\Theta\left[54\Delta\nabla-54J\vartheta+\tfrac{31.27}{25}\Theta^2\right]+24J^2\phi\psi+108\nu\vartheta, \tag{27}$$

which agrees with Wiltheiss' result (Math. Annalen, 29, p. 297) if we make use of the syzygies (233), (323), (226), (224), (336), given by v. Gall in the above-named paper.

3. For the computation of S_4 we need $D(\phi\psi)$ and $D(\nu)$.

a). Since

$$D(\phi_x^3)=M_x^3, \quad D(\psi_x^3)=N_x^3,$$

we have

$$D(\phi_x^3\psi_x^3)=-3\Phi_x^2\,.\,\vartheta_x^4.$$

Hence for $x=u$:

$$D(\phi\psi)=-18\vartheta^2. \tag{28}$$

b). $D(\nu_x^2)=(\nabla_x^2, D(\Delta_x^2))_1+(\Delta_x^2, D(\nabla_x^2))_1=2(\Delta\nabla)^2.\Phi_x^2-(\Phi\nabla)^2.\Delta_x^2-(\Phi\Delta)^2.\nabla_x^2$.

And making use of the relations*

$$(\vartheta, \Delta)_2=\quad(\Delta, \Theta)_1-\tfrac{1}{3}J\Delta,$$
$$(\vartheta, \nabla)_2=-(\nabla, \Theta)_1-\tfrac{1}{3}J\nabla,$$

we obtain for $x=u$, always in Clebsch's notation,

$$D(\nu)=6\nu\Theta+12\mathrm{T}\vartheta+2J\Delta\nabla. \tag{29}$$

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* Caporali, l. c.

The result of the computation of S_4 is

$$S_4 = -\frac{3^6 . 11}{5^3}\Theta^4 + \frac{2^2 . 3^4 . 23}{5}\Theta^2 . \Delta\nabla - 2^2 . 3^4 . (\Delta\nabla)^2 + 2^2 . 3^4 . \Theta^2 . J\vartheta - 2^2 . 3^4 . J^2\vartheta^2$$
$$+ \frac{2^4 . 3^4 . 11}{5}\Theta . \vartheta\nu + \frac{2^5 . 3^2}{5}J^2 . \Theta . \phi\psi - 2^4 . 3^3 J\nu . \phi\psi + 2^4 . 3^4 \mathrm{T}\vartheta^2. \quad (30)$$

4. For the computation of S_5 the value of $D(\mathrm{T})$ is required.

$$\mathrm{T} = (\Delta, \nabla)_2,$$

hence

$$D(\mathrm{T}) = (\Delta_x^2, D(\nabla_x^2))_2 + (D(\Delta_x^2), \nabla_x^2)_2$$
$$= 4\,(\Phi\Delta)(\Phi\nabla)(\Delta\nabla)$$
$$= 24\,(\vartheta\Delta)(\vartheta\nabla)(\Delta\nabla)\,\vartheta_u^2 - 4J\nu_u^2.$$

But*

$$(\vartheta\Delta)(\vartheta\nabla)(\Delta\nabla)\,\vartheta_u^2 - (\vartheta, \nu)_2 = -\tfrac{1}{2}p\pi - \tfrac{1}{6}J\nu,$$

hence

$$D(\mathrm{T}) = -8J\nu - 12p\pi. \quad (31)$$

5. For the computation of S_6 the value of $D(p\pi)$ is required; this can be reduced to $D(\Delta\nabla, \Theta)_2$ as follows:

If we denote $\Delta_x^2\nabla_x^2 = r_x^4$,

we have from (19):

$$r_x^2 r_y^2 = \Gamma_x^2\Gamma_y^2 - J\vartheta_x^2\vartheta_y^2 + \Theta_x^2\Theta_y'^2 - \tfrac{1}{3}(\Theta\Theta')^2(xy)^2. \quad (32)$$

Hence

$$(\Delta\nabla, \Theta)_2 = (\Gamma, \Theta)_2 - J(\vartheta, \Theta)_2 + \tfrac{2}{3}(\Theta\Theta')^2 . \Theta.$$

But from (19b) follows:

$$(\Gamma, \Theta)_2 = -\tfrac{1}{2}p\pi;$$

further,

$$(\vartheta, \Theta)_2 = -\tfrac{1}{2}\nu - \tfrac{1}{3}J\Theta,$$
$$(\Theta\Theta')^2 = \mathrm{T} - \tfrac{1}{2}J^2,$$

hence

$$2(\Delta\nabla, \Theta)_2 = -p\pi + J\nu + \tfrac{4}{3}\mathrm{T}\Theta.$$

Now

$$D(\Delta_x^2\nabla_x^2, \Theta_x^2)_2 = (\Delta_x^2\nabla_x^2, D(\Theta_x^2))_2 + (D(\Delta_x^2\nabla_x^2), \Theta_x^2)_2.$$

From (20) follows:

$$(\Delta_x^2\nabla_x^2, D(\Theta_x^2))_2 = 12\,(\Gamma r)^2\,\Gamma_u^2 r_x^2 - 10J(\vartheta r)^2\,\vartheta_u^2 r_x^2 - \tfrac{1}{3}J^2 r_u^2 r_x^2.$$

Put $x = u$, make use of (32), and observe that†

$$(\Gamma, \Gamma)_2 = \Omega\vartheta, \ (\vartheta, \Gamma)_2 = \tfrac{1}{6}J\Gamma.$$

* Càporali, l. c.

† Berzolari, Rendiconto dell Acc. di Napoli, Serie 2, vol. V, p. 77. Ω is used with the same sign as in Berzolari's and Càporali's papers, viz. $\Omega = -\frac{(p, \pi)_1}{2}$.

Thus we obtain

$$(\Delta\nabla,\, \Gamma)_2 = \Omega\vartheta - \tfrac{1}{2}\Theta\,.\,p\pi - \tfrac{1}{3}\mathrm{T}\Gamma;$$

further,*

$$(\Delta\nabla,\, \vartheta)_2 = -\tfrac{1}{3}J\Delta\nabla - \tfrac{1}{2}\Theta\nu - \tfrac{1}{3}\mathrm{T}\vartheta,$$

hence

$$(\Delta_x^2\nabla_x^2,\, D(\Theta_x^2))_2^{x=u} = 12\Omega\vartheta - 6\Theta p\pi + 5J\nu\,.\,\Theta + \tfrac{10}{3}\mathrm{T}J\vartheta - 4\mathrm{T}\Gamma + 3J^2\Delta\nabla.$$

On the other hand,

$$(D(\Delta_x^2\nabla_x^2),\, \Theta_x^2)_2 = 2(\nu_x^2\Phi_x^2,\, \Theta_x^2)_2 = \Phi_x^2(\nu,\, \Theta)_2 + \nu_x^2(\Phi,\, \Theta)_2 - \tfrac{2}{3}(\Phi\nu)^2.\,\Theta_x^2.$$

Observe that by definition

$$(\nu,\, \Theta)_2 = \Omega;$$

further

$$(\Phi,\, \Theta)_2 = -3\nu - J\Theta, \quad (\Phi,\, \nu)_2 = 3p\pi + 2J\nu.$$

Putting $x = u$ we have therefore

$$(D(\Delta_x^2\nabla_x^2),\, \Theta_x^2)_2^{x=u} = 6\Omega\vartheta - 3\nu^2 - \tfrac{7}{3}J\Theta\nu - 2\Theta p\pi.$$

And if we make use of the relation†

$$2\nu^2 = -4\Omega\vartheta + 2\Delta\nabla(2\mathrm{T} - J^2) - 4\mathrm{T}\Theta^2 + 4\Theta p\pi - J\nu\Theta$$

we finally obtain

$$D(p\pi) = -24\Theta\,.\,p\pi - 12\Omega\vartheta + 8J^2.\,\Delta\nabla + 8J\nu\Theta + 16\,TJ\vartheta. \tag{34}$$

6. For the computation of S_7 the value of $D(\Omega)$ is required. Since

$$\Omega = (\Theta,\, \nu)_2,$$

we have

$$D(\Omega) = (\Theta_x^2,\, D(\nu_x^2))_2 + (D(\Theta_x^2),\, \nu_x^2)_2.$$

The value previously obtained for $D(\nu_x^2)$ can easily be transformed into

$$D(\nu_x^2) = 2J\Gamma_x^2\Gamma_u^2 - 2J^2\vartheta_u^2\vartheta_x^2 + 2J\Theta_u^2\Theta_x'^2 + 12\mathrm{T}\vartheta_u^2\vartheta_x^2 + 6\nu_u^2\Theta_x^2 - (6\Omega - \tfrac{1}{3}J^3 - 2J\mathrm{T})(ux)^2$$

by means of the identity

$$\Delta_u^2\nabla_x^2 + \nabla_u^2\Delta_x^2 = 2\Gamma_u^2\Gamma_x^2 - 2J\vartheta_u^2\vartheta_x^2 + 2\Theta_u^2\Theta_x'^2 + \tfrac{1}{3}J^2(ux)^2.$$

Hence follows, by using previous results,

$$(\Theta_x^2,\, D(\nu_x^2))_2 = -Jp\pi - 2\nu J^2 - 6\Omega\Theta.$$

On the other hand we have, on account of (20),

$$(D(\Theta_x^2),\, \nu_x^2)_2 = 12(\Gamma\nu)^2\Gamma_u^2 - 10J(\vartheta\nu)^2\vartheta_u^2 - \tfrac{1}{3}J^2\nu_u^2.$$

* Berzolari, l. c., p. 73.

† Cf. v. Gall, l. c., the syzygies $(444)_2$, and A) on p. 426, together with Berzolari, l. c., p. 74, (4).

But
$$(\vartheta, \nu)_2 = \tfrac{1}{2} p\pi + \tfrac{1}{6} J\nu,$$
$$(H_\vartheta, \nu)_2 = -\tfrac{1}{36} J^2\nu + \tfrac{1}{6} Jp\pi - \tfrac{1}{2}\Omega\Theta;$$

hence since: $3\Gamma = 6H_\vartheta + J\vartheta$,
$$(\Gamma, \nu)_2 = \tfrac{1}{2} Jp\pi - \Omega\Theta.$$

Thus we obtain
$$D(\Theta_x^2, \nu_x^2)_2 = Jp\pi - 2J^2\nu - 12\Omega\Theta$$

and
$$D(\Omega) = -4J^2\nu - 18\Omega\Theta. \tag{35}$$

Since $D(\Omega)$ *contains no new simultaneous concomitants of* ϕ *and* ψ, it follows that the terms following upon S_6 contain no other concomitants but those which have already made their appearance in the first six terms, and we have thus proved Wiltheiss-Brioschi's result:

Theorem V.

The successive terms in the expansion

$$\sigma_{\phi\psi}(u_1, u_2) = 1 + \frac{S_1}{2!} + \frac{S_2}{4!} + \frac{S_3}{6!} + \dots$$

are expressible as integral functions of the following nine simultaneous concomitants of $\phi(u)$ *and* $\psi(u)$:

$$\begin{array}{llll} \phi\psi, & \vartheta = (\phi, \psi)_1, & \Theta = (\phi, \psi)_2, & J = (\phi, \psi)_3, \\ \Delta\nabla, & & \nu = (\nabla, \Delta)_1, & \mathrm{T} = (\Delta, \nabla)_2, \\ p\pi, & & \Omega = -\dfrac{(p, \pi)_1}{2} & \end{array}$$

and the effect of the operator D upon these forms is exhibited in the following table:

$$\begin{aligned}
D(\Theta) &= 12\Delta\nabla - 12\Theta^2 + 2J\vartheta, \\
D(J) &= -18\nu - 6J\Theta, \\
D(\vartheta) &= 4J.\phi\psi - 18\Theta\vartheta, \\
D(\Delta\nabla) &= 12\nu\vartheta, \\
D(\phi\psi) &= -18\vartheta^2, \\
D(\nu) &= 6\nu\Theta + 12\mathrm{T}\vartheta + 2J\Delta\nabla, \\
D(\mathrm{T}) &= -8J\nu - 12p\pi, \\
D(p\pi) &= -24\Theta.p\pi - 12\Omega\vartheta + 8J^2.\Delta\nabla + 8J\nu.\Theta + 16\mathrm{T}J\vartheta, \\
D(\Omega) &= -4J^2\nu - 18\Omega\Theta.
\end{aligned}$$

The above results can be compared with Wiltheiss' results (Math. Ann., Bd. 36, p. 153) as follows:

We obtain easily

$$(\Phi\phi)\,\Phi_x\phi_x^2 = 3\Theta_u^2 \cdot \phi_x^3 - 12\Theta_u\Theta_x \cdot \phi_u\phi_x^2 + 6\Theta_x^2 \cdot \phi_u^2\phi_x + 3\Delta_u^2 \cdot \psi_x^3 + 2J\phi_u\phi_x^2\,(xu).$$

Hence it follows that Brioschi's operator D is expressible in terms of Wiltheiss' operators δ_1, δ_2, δ_3, δ_4, δ_5 as follows:

$$D = 3\,[3\delta_1 - 12\delta_2 + 6\delta_3 + 3\delta_4 + 2\delta_5]_{x=u}.$$

Our results agree exactly with Wiltheiss' results, whereas the values for $D(J)$, $D(T)$, $D(p\pi)$, $D(\Omega)$ do not agree with Brioschi's results.

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